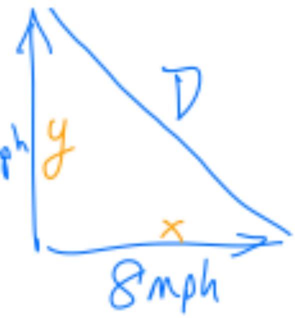
4.65 Monique and Alice eat dinner together and then go their separate ways. Alice leaves the restaurant and travels due north at a rate of 35 miles per hour. Monique jogs home to the east at a rate of 8 miles per hour. After fifteen minutes, Alice smiles to herself and dreams of questions: how fast is the distance between Alice and Monique increasing, and how far apart are they? Help Alice answer these questions.



$\frac{1}{4}\cdot 35 = \frac{35}{4}$ miles $= 8.75$ mi $= y$

15 min $= \frac{1}{4}$ hr.

$\frac{1}{4}(8) = 2$ miles $= x$

Given $\frac{dx}{dt} = 8$ mi/hr.

$\frac{dy}{dt} = 35$ mi/hr

$\frac{dD}{dt} = ?$

$$x^2 + y^2 = D^2$$

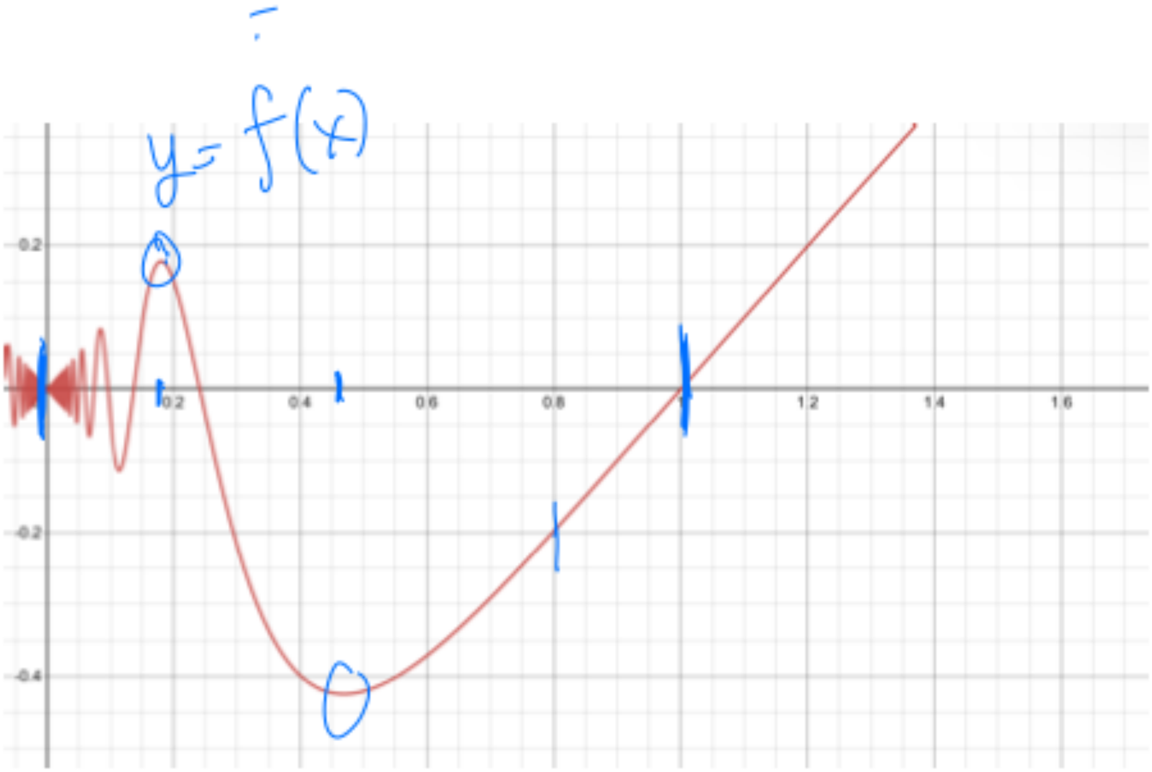
derivative:

$$\cancel{2}\underset{2\text{ mi}}{x}\underset{8\text{ mi/hr}}{\frac{dx}{dt}} + \cancel{2}\underset{8.75\text{ mi}}{y}\underset{35\text{ mi/hr}}{\frac{dy}{dt}} = \cancel{2}\underset{\sqrt{4+(8.75)^2}}{D}\frac{dD}{dt}$$

Interesting functions & graphs

$$f(x) = x\sin\left(1 - \frac{1}{x}\right) \quad \text{on } (0,1)$$

$$g(x) = x\sin\left(\frac{1}{1-x}\right) \quad \text{on } (0,1)$$



Global max of f(x) on (0,1) $x \approx 0.18$

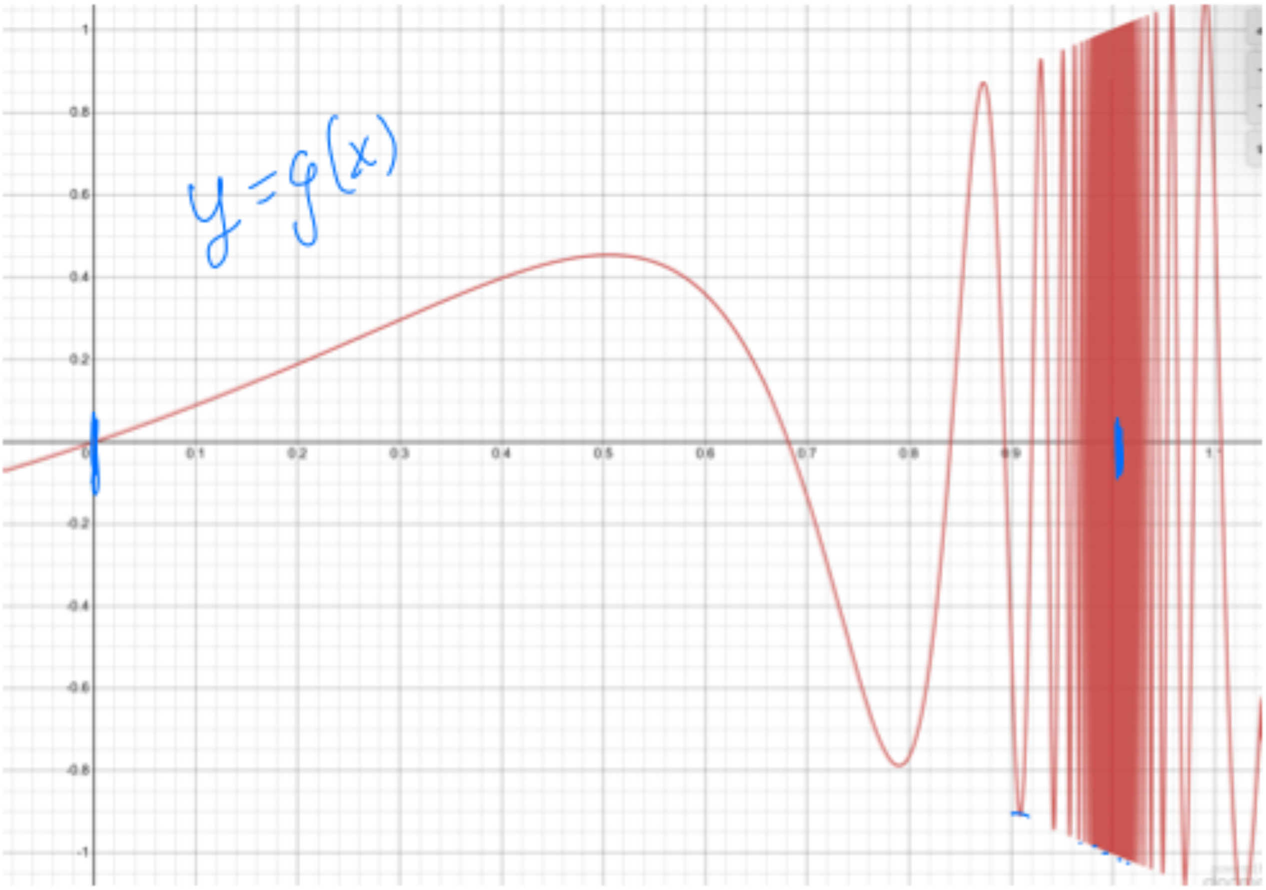
max value $= \approx 0.175$

Global min of f(x) on (0,1) $x \approx 0.45$

min value $\approx -.42$

Notice there are an infinite # of critical pts!

(They are all local max's & local min's)



Global min & max of g(x) on (0,1)

None exist!

An infinite number of critical pt

(all local mins & local max's)

let $F(x) = \sum_{n=0}^{\infty} 2^{-n} \cos(3^n x)$

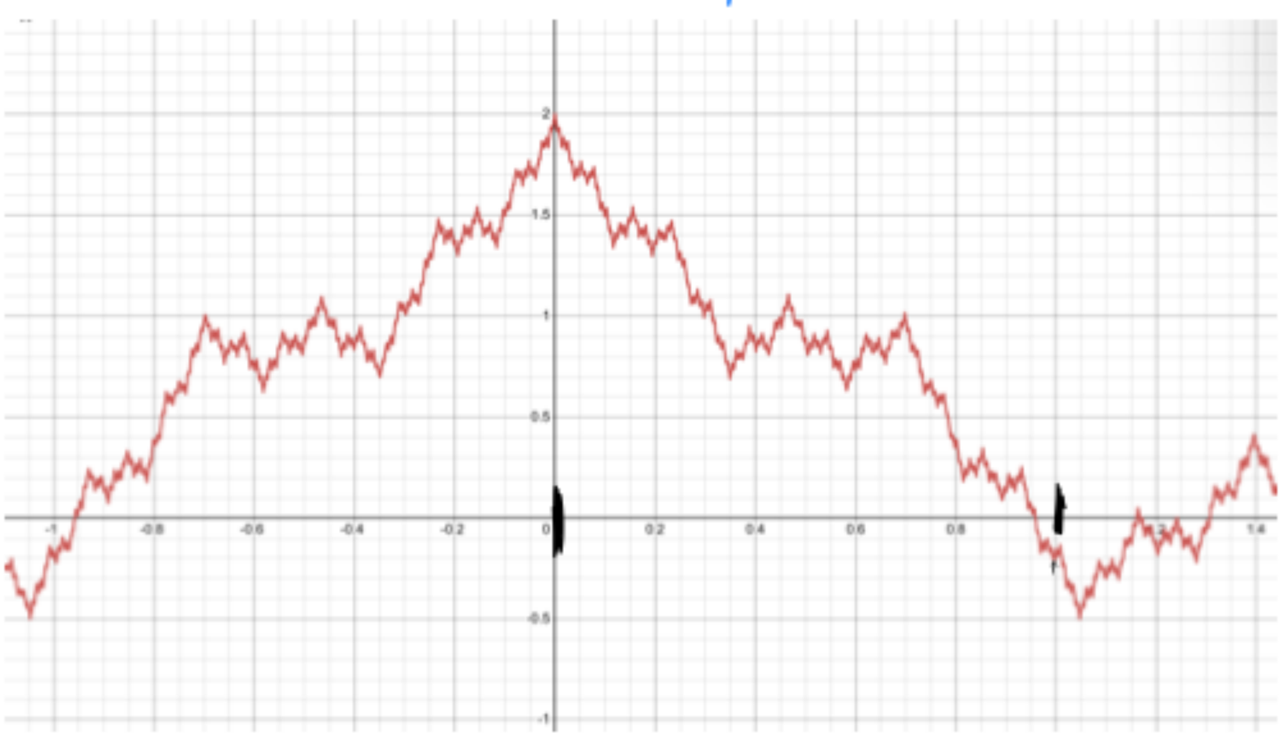
summation sign (add)

$$= 2^{-0}\cos(3^0 x) + 2^{-1}\cos(3^1 x) + 2^{-2}\cos(3^2 x)$$

$$+ 2^{-3}\cos(3^3 x) + 2^{-4}\cos(3^4 x) + 2^{-5}\cos(3^5 x)$$

$$+ \dots\dots$$

— This makes a function that is continuous at every point but is differentiable at no x.



What's the global max & min of $F(x)$ on the interval $[0,1]$.

(has to exist, because F is a continuous fcn on a closed interval. Extreme Value Thm guarantees that we have a global max & min.

global max: $x=0$, max value $=2$

global min: $x \approx 0.98$, min. value $\simeq -0.22$

Real World Questions:

① Eric's Popsicle Syrup company needs to design a cylindrical can to hold 32oz of syrup. The metal can has a top & bottom that cost 21¢ per square inch to make, and the cylindrical side costs 17¢ per square inch, Engineer Luke figures out how to design the can so that it costs the least amount of money. The business manager

Samiksha is very happy and gives Luke a gold star, made out of real gold. How did Luke design the can?

$V = 32oz$
$= \pi r^2 h$

minimize

$$\text{Cost} = P(\ ) = \left(\frac{21¢}{in^2}\right) 2\pi r^2 + \left(\frac{17¢}{in^2}\right) 2\pi r h$$

top + bottom.

side area

$V = 32 = \pi r^2 h$

$h = \frac{32oz}{\pi r^2}$

$$\text{Cost} = (21)(2\pi r^2) + (17)\left(2\cancel{\pi} r\left(\frac{32oz}{\cancel{\pi} r^2}\right)\right)$$

$$C(r) = 42\pi r^2 ¢ + \frac{1088\ oz}{r\ in} ¢ \cdot \frac{1\ in^3}{0.55 oz}$$

inches

Need oz in in^3: $1\ in^3 = 0.55\ oz$

$$\Rightarrow C(r) = 42\pi r^2 + \frac{1978}{r}$$

want the global minimum of this.

<u>interval</u>: $0 < r < \infty$ $\quad (0, \infty)$

(Need cr pts in inside of interval + limits as $r \to 0^+$, $r \to \infty$.)

$$C'(r) = 84\pi r - \frac{1978}{r^2} = 0$$

$$84\pi r - \frac{1978}{r^2} = 0$$

$$84\pi r = \frac{1978}{r^2}$$

multiply by r^2
divide by 84π

$$r^3 = \frac{1978}{84\pi}$$

$$r = \sqrt[3]{\frac{1978}{84\pi}} = 1.957$$

$$\boxed{1.96 \text{ in}}$$

Check

$$\lim_{r \to 0^+} C(r) = \lim_{r \to 0^+}\left(\underbrace{42\pi r^2}_{\to 0} + \underbrace{\frac{1978}{r}}_{\to \infty}\right) = \infty$$

$$\lim_{r \to \infty} C(r) = \lim_{r \to \infty}\left(\underbrace{42\pi r^2}_{\to \infty} + \underbrace{\frac{1978}{r}}_{\to 0}\right) = \infty$$

$\boxed{r = 1.96 \text{ in must be the global min!}}$

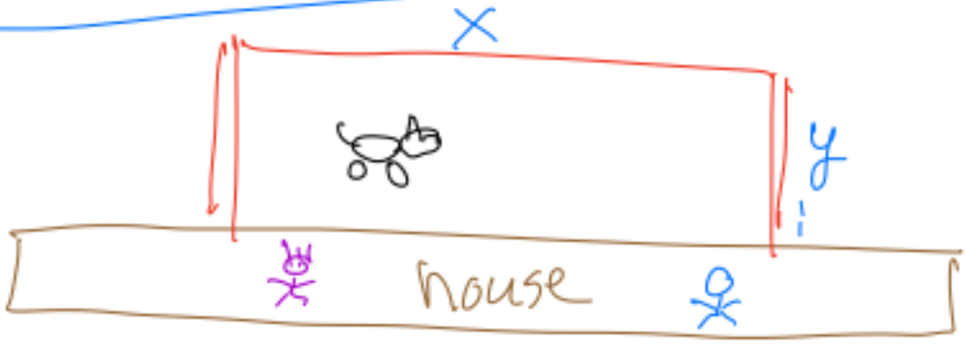
Need h: $h = \frac{32\,oz}{\pi r^2} = \frac{32\,oz}{\pi (1.96)^2} \cdot \frac{1\,in^3}{0.55\,oz} = \boxed{4.82\,in}$

Luke's design was a cylindrical can with $r = 1.96$ in, $h = 4.82$ in.

General ideas of solving max/min (optimization) problems:

1. Draw picture with variables.
2. The quantity that you are trying to minimize or maximize is the function.
3. Using info in the question, put the function in terms of one variable.
4. Find the interval for that variable, based on the given information.
5. Then use derivative & limits to find the global max or min.

$\boxed{\text{Example}}$ There is a big straight wall on the back of Dat's house. Dat's roommate Regan has a dog named Monica, who is very nice. Reagan & Dat want to make a rectangular fenced area on the back of the house. They have to use twice as much fence on the sides as on the side parallel to the house. They have a total of 200 ft of fence (but they'll have to double it on the sides). How should they design the area so that Monica has the most space to run?



Function: Area $= A = xy$

given $200\text{ ft} = x + 4y \Rightarrow x = 200 - 4y$

$$A(y) = (200-4y)y$$

want max: $A(y) = 200y - 4y^2$

$$0 < y \le 50$$

interval:

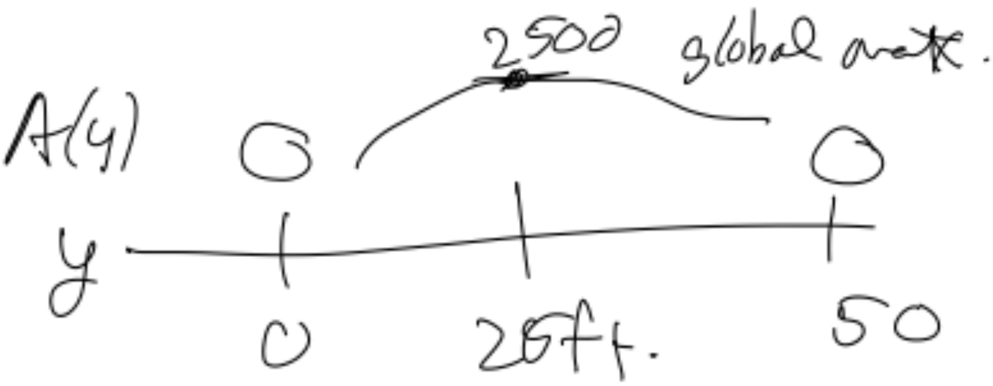
Cr pts + $y \to 0^+$, $y = 50$

$$A'(y) = 200 - 8y = 0$$

$$y = \frac{200}{8} = 25 \text{ ft.}$$

$$\lim_{y\to 0^+}(\underbrace{200y}_{0} - \underbrace{4y^2}_{0}) = 0.$$

$$A(50) = 200(50) - 4(50)^2 = 0.$$



$$A(25) = (200)(25) - 4\underset{625}{(25)^2}$$

$$= 5000 - 2500$$

$$= \boxed{2500\ ft^2}$$

$$y = 25\ ft$$

$$x + 4y = 200$$

$$\Rightarrow \quad x = 200 - 4y =$$
$$200 - 4(25) = 100$$